Computational Testing of Scarf Algorithm for Near Feasible Stable Matching with Couples

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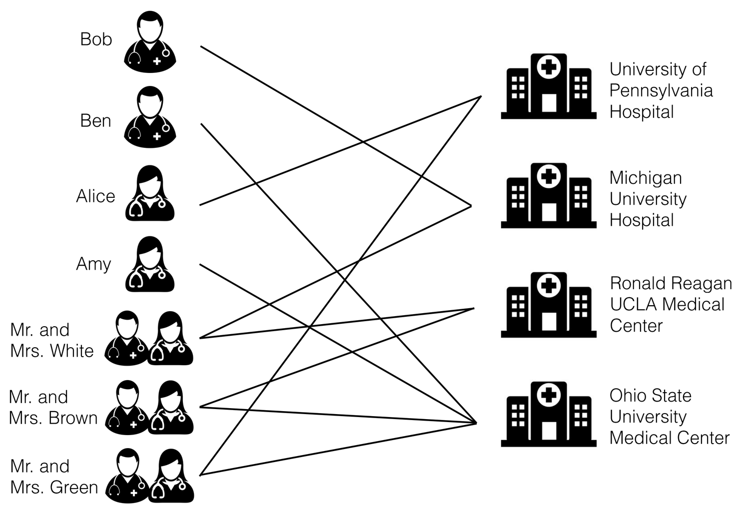
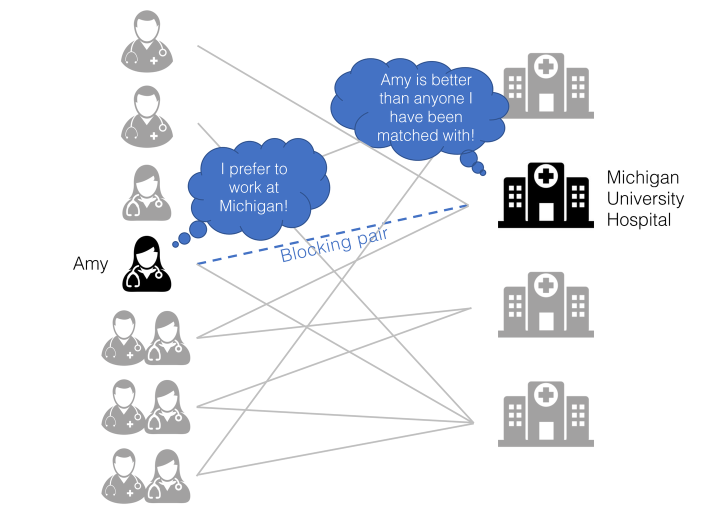
Introduction

- Since Gale and Shapley [1], matching theory is a central topic of market design
- The objective is to matching one side with other side in a two-sided market, e.g. matching doctors to hospitals, such that some stability constraint is satisfied.
- Complementarity in preferences (e.g. joint preference of couple doctors) becomes an important aspect in many real world instances.

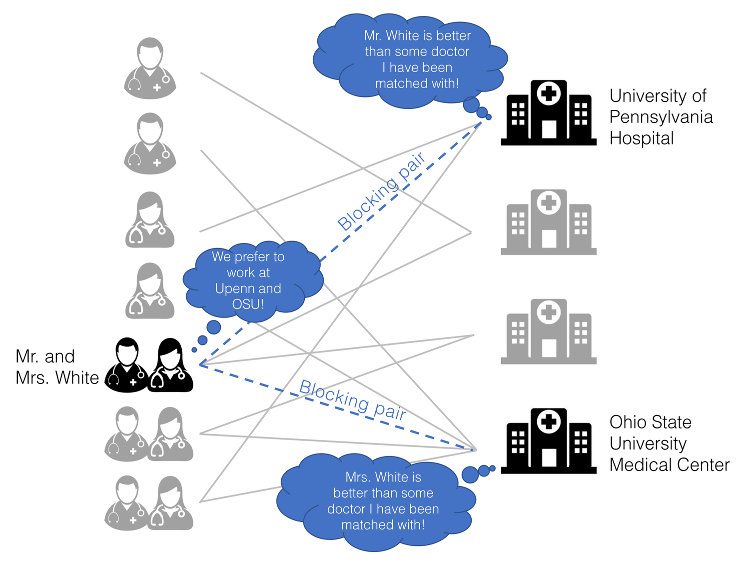


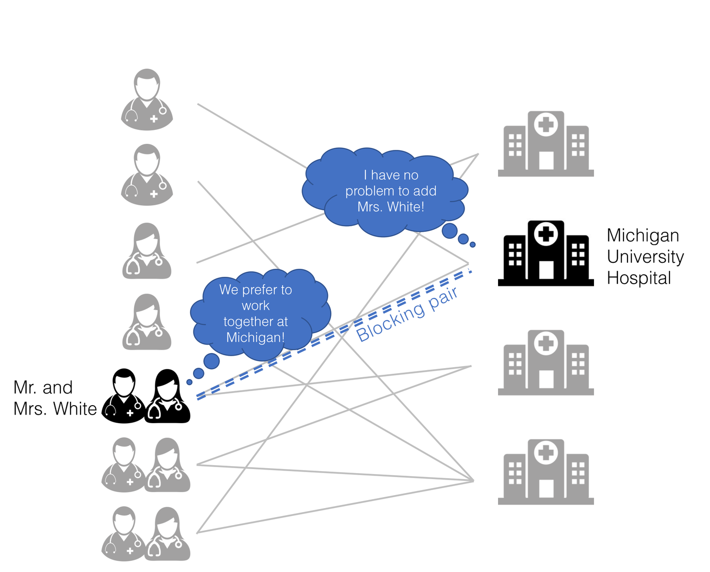


Model Setup

A pair of hospitals and doctors is said to block a matching μ if

1. A pair $s \in D^1$ and $h \in H$ can block μ if $h \succ_s \mu(s)$ and $s \in ch_h(\mu(h) \cup s)$.
2. A triple $(c, h, h') \in D^2 \times (H \cup \{\emptyset\}) \times (H \cup \{\emptyset\})$ with $h \neq h'$ can block μ if $(h, h') \succ_c \mu(c)$, $f_c \in ch_h(\mu(h) \cup f_c)$ when $h \neq \emptyset$ and $m_c \in ch_{h'}(\mu(h') \cup m_c)$ when $h' \neq \emptyset$.
3. A pair $(c, h) \in D^2 \times H$ can block μ if $(h, h) \succ_c \mu(c)$ and $(f_c, m_c) \subseteq ch_h(\mu(h) \cup c)$.





Given preference lists for single doctors and couples; a matching μ is **stable with respect to a capacity vector** k if μ cannot be blocked in any of the three ways listed above.
The basic setting of Scarf's algorithm is a linear programming problem:

$$\begin{aligned} &\sum_{h \in \hat{H}} x_{(d,h)} = 1, \quad \forall d \in D^1 \\ &\sum_{h_1, h_2 \in \hat{H}} x_{(c,h)} = 1, \quad \forall c \in D^2 \\ &x_{(\emptyset_d, h)} + \sum_{d \in D^1} x_{(d,h)} + \sum_{c \in D^2} \left(\sum_{\substack{h' \in \hat{H} \\ h' \neq h}} x_{(c,h,h')} + \sum_{\substack{h' \in \hat{H} \\ h' \neq h}} x_{(c,h',h)} + 2x_{(c,h,h)} \right) = k_h, \quad \forall h \in H \end{aligned} \tag{1}$$

The above defines a system $Ax = b$. Additionally, every row of A is associated with an ordering $\succ_i$ of its supporting columns.

Theorem 1 (Scarf's lemma)**.** *There exists an extreme point of* $\{x \in \mathbb{R}^N_+ : Ax = b\}$ *that dominates every column of* A *[4], i.e.* $\forall$ col $j, \exists$ row i *such that* $k \succeq_i j \ \forall k$ s.t. $x^*_k > 0$. *Furthermore, one of such extreme points can be found by Scarf algorithm.*

Choosing a suitable $\succ_i$ for each row, the condition of domination could translate to the condition of stability for the polytope defined by (1). Hence a procedure to find a stable matching is described as follows.

- Use Scarf's algorithm to find such an extreme point.
- If the extreme point is integral, we have found a stable matching.
- If the extreme point is fractional, we apply a rounding procedure described in [3] to obtain a near feasible stable matching with overallocation at some hospitals.

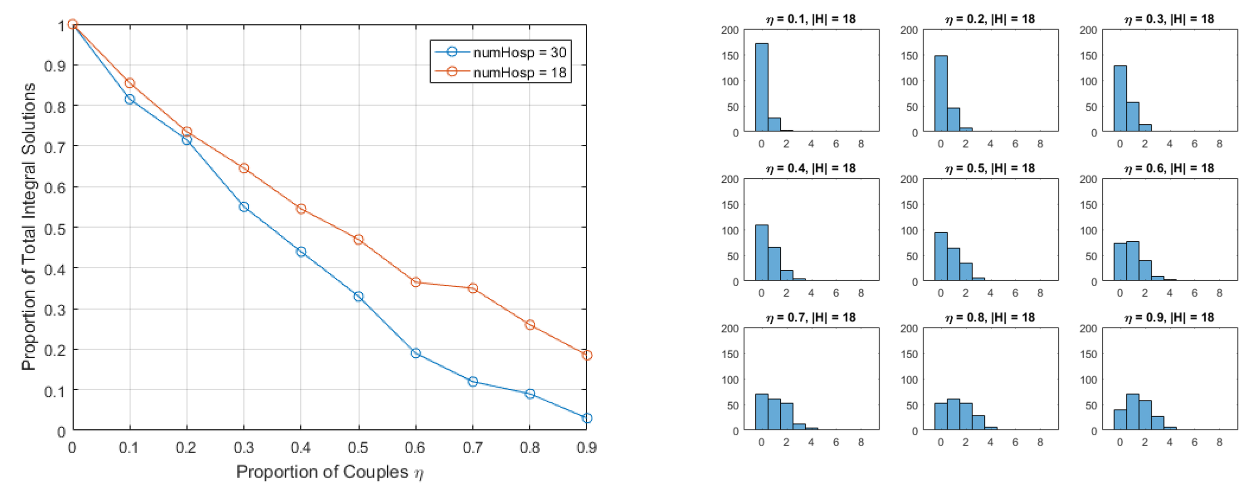
Main Result

Theorem 2. *Given a stable matching problem of doctors and hospitals with complementarities, if the couples exhibit weakly responsive [2] preferences, a stable matching exists and can be found by Nguyen and Vohra's [3] construction along with Scarf's algorithm [4].*

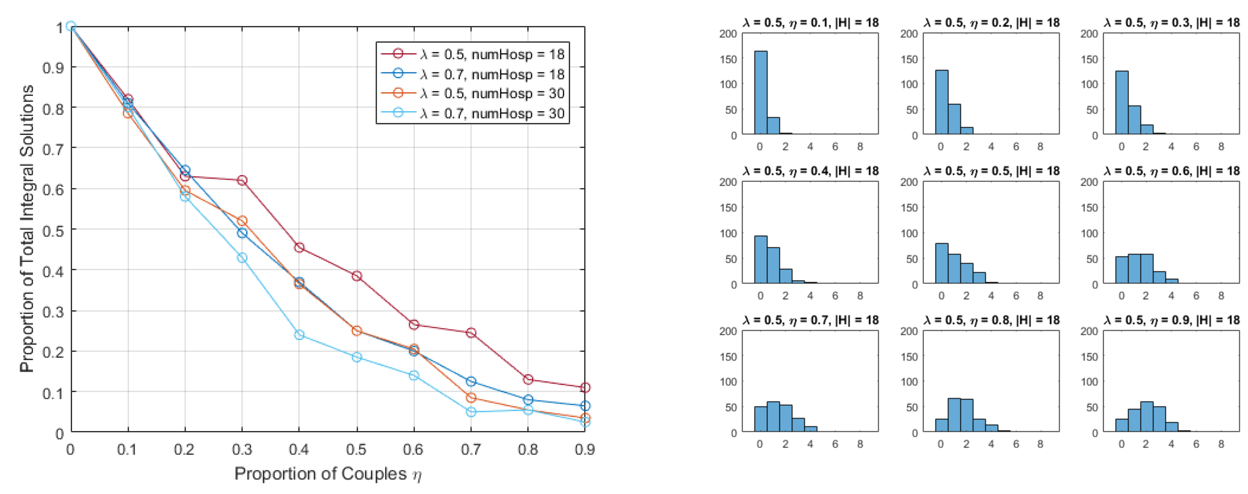
Proof Outline:

1. Show that the basis returned by Scarf's algorithm has a special structure
2. Apply basic row and column manipulation on the sub-matrix indexed by the basis, finally arrived at a network matrix, which we know is unimodular.
3. Since basic row and column manipulation preserves determinant, we conclude that the basis submatrix has determinant ± 1. Hence the extreme point is integral.

We also conduct simulations of the algorithm on two random generators that each yield preferences that are not weakly responsive.
The following figure shows empirical results for uniform random instance generators. The figure in the left reports the fraction of integral solutions we obtain from 200 simulation runs for the case of 18 and 30 hospitals respectively. The figure in the right displays a histogram for over(under)-allocation for a single hospital.



For the region-based random instance generator we performed the same experiment.



Conclusions

- We show that if the couples' preferences are weakly responsive, then Scarf's algorithm always produces an integer solution, i.e., a stable matching.
- The technique used in the proof also suggests a new way to identify cases that have a stable matching via Scarf's algorithm: showing that the dominating extreme points (rather than all extreme points) are integral.
- We investigated the computational issues of recent algorithm that uses Scarf's algorithm followed by rounding of any non-integer solution. These tests shows that the performance of the algorithm is significantly better than the worst case theoretical bounds.
- Our analysis shows that the use of Scarf's lemma is practical for the problem of matching with couples.

Acknowledgements

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